

Local Proof of Algebraic Characterization of Free Actions^{*}

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Abstract. Let G be a compact Hausdorff topological group acting on a compact Hausdorff topological space X . Within the C^* -algebra $C(X)$ of all continuous complex-valued functions on X , there is the Peter–Weyl algebra $\mathcal{P}_G(X)$ which is the (purely algebraic) direct sum of the isotypical components for the action of G on $C(X)$. We prove that the action of G on X is free if and only if the canonical map $\mathcal{P}_G(X) \otimes_{C(X/G)} \mathcal{P}_G(X) \rightarrow \mathcal{P}_G(X) \otimes \mathcal{O}(G)$ is bijective. Here both tensor products are purely algebraic, and $\mathcal{O}(G)$ denotes the Hopf algebra of “polynomial” functions on G .

Key words: compact group; free action; Peter–Weyl–Galois condition

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With admiration and affection, to Marc A. Rieffel on the occasion of his 75th birthday

1 Theorem

Given a compact Hausdorff topological group G , we denote by $\mathcal{O}(G)$ the dense Hopf $*$ -subalgebra of the commutative C^* -algebra $C(G)$ spanned by the matrix coefficients of irreducible representations of G . Let X be a compact Hausdorff topological space with a given continuous (right) action of G . The action map

$$X \times G \ni (x, g) \longmapsto xg \in X$$

determines a map of C^* -algebras

$$\delta: C(X) \longrightarrow C(X \times G).$$

Moreover, denoting by \otimes the purely algebraic tensor product over the field \mathbb{C} of complex numbers, we define the *Peter–Weyl subalgebra* [2, (3.1.4)] of $C(X)$ as

$$\mathcal{P}_G(X) := \{a \in C(X) \mid \delta(a) \in C(X) \otimes \mathcal{O}(G)\}.$$

Using the coassociativity of δ , one can check that $\mathcal{P}_G(X)$ is a right $\mathcal{O}(G)$ -comodule algebra. In particular, $\mathcal{P}_G(G) = \mathcal{O}(G)$. The assignment $X \mapsto \mathcal{P}_G(X)$ is functorial with respect to continuous

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G -equivariant maps and comodule algebra homomorphisms. We call it the *Peter–Weyl functor*. Equivalently [10, Proposition 2.2], $\mathcal{P}_G(X)$ is the (purely algebraic) direct sum of the isotypical components for the action of G on $C(X)$ (see [7, p. 31] and [6], cf. [8, Theorem 1.5.1]). In the special case that the action of G on X is free, $\mathcal{P}_G(X)$ is the algebra of all continuous sections of the vector bundle E on X/G , where

$$E := X \times_G \mathcal{O}(G). \quad (1)$$

Note that in forming this vector bundle, $\mathcal{O}(G)$ is topologized as the direct limit of its finite-dimensional vector subspaces, not by the norm topology.

The theorem of this paper is:

Theorem 1. *Let X be a compact Hausdorff space equipped with a continuous (right) action of a compact Hausdorff group G . Then the action is free if and only if the canonical map*

$$\begin{aligned} \text{can}: \mathcal{P}_G(X) \otimes_{C(X/G)} \mathcal{P}_G(X) &\longrightarrow \mathcal{P}_G(X) \otimes \mathcal{O}(G), \\ \text{can}: x \otimes y &\longmapsto (x \otimes 1)\delta(y), \end{aligned} \quad (2)$$

is bijective. (Here both tensor products are purely algebraic.)

Definition 1. The action of a compact Hausdorff group G on a compact Hausdorff space X satisfies the *Peter–Weyl–Galois (PWG) condition* iff the canonical map (2) is bijective.

Our result states that the usual formulation of free action is equivalent to the algebraic PWG-condition. In particular, our result provides a framework for extending Chern–Weil theory beyond the setting of differentiable manifolds and into the context of cyclic homology – non-commutative geometry [3].

2 Proof

The proof of the equivalence of freeness and the PWG-condition consists of six steps. The first step takes care of the easy implication of the equivalence, and the remaining five steps prove the difficult implication of the equivalence.

2.1 PWG-condition \Rightarrow freeness

It is immediate that the action is free, i.e. $xg = x \implies g = e$ (where e is the identity element of G), if and only if

$$\begin{aligned} F: X \times G &\longrightarrow X \times_{X/G} X, \\ F: (x, g) &\longmapsto (x, xg), \end{aligned}$$

is a homeomorphism. Here $X \times_{X/G} X$ is the subset of $X \times X$ consisting of pairs (x_1, x_2) such that x_1 and x_2 are in the same G -orbit.

This is equivalent to the assertion that the $*$ -homomorphism

$$F^*: C\left(X \times_{X/G} X\right) \longrightarrow C(X \times G)$$

obtained from the above map F is an isomorphism. Note that F is always surjective, so that the $*$ -homomorphism F^* is always injective. Furthermore, there is the following commutative diagram in which the vertical arrows are the evident maps:

$$\begin{array}{ccc}
 \mathcal{P}_G(X) \underset{C(X/G)}{\otimes} \mathcal{P}_G(X) & \xrightarrow{\text{can}} & \mathcal{P}_G(X) \otimes \mathcal{O}(G) \\
 \downarrow & & \downarrow \\
 C\left(X \underset{X/G}{\times} X\right) & \xrightarrow{F^*} & C(X \times G).
 \end{array} \tag{3}$$

Since the right-hand side of the canonical map (2), i.e. $\mathcal{P}_G(X) \otimes \mathcal{O}(G)$, is dense in the C^* -algebra $C(X \times G)$, validity of the PWG-condition combined with the commutativity of the diagram (3) implies that the image of the $*$ -homomorphism F^* is dense in $C(X \times G)$. Therefore, as the image of a $*$ -homomorphism of C^* -algebras is always closed, PWG implies surjectivity of F^* , which in turn implies that the action of G on X is free.

2.2 Reduction to surjectivity and matrix coefficients

Lemma 1. *Let X be a compact Hausdorff space equipped with a continuous (right) action of a compact Hausdorff group G . Then the canonical map is surjective if and only if for any matrix coefficient h of an irreducible representation of G , the element $1 \otimes h$ is in the image of the canonical map. Moreover, if the canonical map is surjective, then it is bijective.*

Proof. First observe that the canonical map is a homomorphism of left $\mathcal{P}_G(X)$ -modules. The first assertion of the lemma follows by combining this observation with the fact that matrix coefficients of irreducible representations span $\mathcal{O}(G)$ as a vector space.

The Hopf algebra $\mathcal{O}(G)$ is cosemisimple. Hence, by the result of H.-J. Schneider [9, Theorem I], if the canonical map is surjective, then it is bijective. \blacksquare

Alternately, assuming that the action of G on X is free, injectivity can be directly proved by using the vector-bundle point of view indicated in (1) (see [1]).

2.3 Reduction to free actions of compact Lie-groups

Assume that Theorem 1 holds for compact Lie groups. In this section, we prove that this special case implies Theorem 1 in general.

Let $\varphi: G \rightarrow U(n)$ be any finite-dimensional representation of G . Set

$$X_\varphi := X \underset{G}{\times} U(n).$$

Thus $X_\varphi = (X \times U(n))/G$, where G acts on $X \times U(n)$ by $(x, u)g := (xg, \varphi(g^{-1})u)$. The group $U(n)$ acts on X_φ by $[(x, u)]v := [(x, uv)]$, and this action is free. The map

$$\Phi: X \longrightarrow X_\varphi, \quad x \longmapsto [(x, I_n)],$$

where $I_n \in U(n)$ is the identity matrix, has the equivariance property

$$\Phi(xg) = \Phi(x)\varphi(g). \tag{4}$$

Hence Φ and φ induce maps $\Phi^*: \mathcal{P}_{U(n)}(X_\varphi) \rightarrow \mathcal{P}_G(X)$ and $\varphi^*: \mathcal{O}(U(n)) \rightarrow \mathcal{O}(G)$. The equivariance property (4) implies commutativity of the diagram

$$\begin{array}{ccc} \mathcal{P}_{U(n)}(X_\varphi) \otimes_{C(X/G)} \mathcal{P}_{U(n)}(X_\varphi) & \xrightarrow{\text{can}} & \mathcal{P}_{U(n)}(X_\varphi) \otimes \mathcal{O}(U(n)) \\ \downarrow \Phi^* \otimes \Phi^* & & \downarrow \Phi^* \otimes \varphi^* \\ \mathcal{P}_G(X) \otimes_{C(X/G)} \mathcal{P}_G(X) & \xrightarrow{\text{can}} & \mathcal{P}_G(X) \otimes \mathcal{O}(G). \end{array}$$

Therefore surjectivity of the upper canonical map implies that $1 \otimes h$ is in the image of the lower canonical map, where h is any matrix coefficient of φ . By Lemma 1, this implies the PWG-condition.

2.4 Local triviality for free actions of compact Lie groups

We recall the theorem of A.M. Gleason:

Theorem 2 ([4]). *Let G be a compact Lie group acting freely and continuously on a completely regular space X . Then X is a locally trivial G -bundle over X/G .*

Combining the Gleason theorem with Section 2.3, we infer that the PWG-condition is valid for free actions if it is valid for locally trivial free actions.

2.5 Reduction to the trivial-bundle case

Assume that the action of G on X is free and locally trivial. Since the quotient space X/G is compact Hausdorff, we can choose a finite open cover U_1, \dots, U_r of X/G such that each $\pi^{-1}(U_j)$ is a trivializable principal G -bundle over U_j . Here $\pi: X \rightarrow X/G$ is the quotient map. On X/G , let ψ_1, \dots, ψ_r be a partition of unity subordinate to the cover U_1, \dots, U_r . Then, for each j there is the canonical map

$$\text{can}_j: \mathcal{P}_G(\pi^{-1}(\text{supp}(\psi_j))) \otimes_{C(\text{supp}(\psi_j))} \mathcal{P}_G(\pi^{-1}(\text{supp}(\psi_j))) \longrightarrow \mathcal{P}_G(\pi^{-1}(\text{supp}(\psi_j))) \otimes \mathcal{O}(G).$$

Assume that for each $j \in \{1, \dots, r\}$ there exist elements

$$p_{j1}, q_{j1}, \dots, p_{jn}, q_{jn} \in \mathcal{P}_G(\pi^{-1}(\text{supp}(\psi_j))) \quad \text{such that} \quad \text{can}_j \left(\sum_{i=1}^n p_{ji} \otimes q_{ji} \right) = 1 \otimes h.$$

Let \widetilde{p}_{ji} 's and \widetilde{q}_{ji} 's be functions on X obtained respectively from functions p_{ji} 's and q_{ji} 's by zero-value extension. Then for each i and j we take

$$\widetilde{p}_{ji} \sqrt{\psi_j \circ \pi}, \quad \widetilde{q}_{ji} \sqrt{\psi_j \circ \pi} \in \mathcal{P}_G(X),$$

and for any $x \in X$ and $g \in G$, using the commutativity of the diagram (3), we obtain

$$\begin{aligned}
& \text{can} \left(\sum_{j=1}^r \sum_{i=1}^n \widetilde{p}_{ji} \sqrt{\psi_j \circ \pi} \otimes \widetilde{q}_{ji} \sqrt{\psi_j \circ \pi} \right) (x, g) \\
&= \left(\sum_{j=1}^r \sum_{i=1}^n \widetilde{p}_{ji} \sqrt{\psi_j \circ \pi} \otimes \widetilde{q}_{ji} \sqrt{\psi_j \circ \pi} \right) (x, xg) \\
&= \sum_{\text{all } j \text{ s.t. } \pi(x) \in U_j} (\psi_j \circ \pi)(x) \sum_{i=1}^n p_{ji}(x) q_{ji}(xg) \\
&= \sum_{\text{all } j \text{ s.t. } \pi(x) \in U_j} (\psi_j \circ \pi)(x) \text{can}_j \left(\sum_{i=1}^n p_{ji} \otimes q_{ji} \right) (x, g) \\
&= \sum_{j=1}^r (\psi_j \circ \pi)(x) h(g) \\
&= (1 \otimes h)(x, g).
\end{aligned}$$

Hence validity of the PWG-condition in the trivial-bundle case implies that the PWG-condition holds for actions that are free and locally trivial.

2.6 The trivial-bundle case

First note that

$$\mathcal{P}_G(Y \times G) = C(Y) \otimes \mathcal{O}(G). \quad (5)$$

This is implied by two facts: (1) quite generally $\mathcal{P}_G(X) \subseteq C(X)$ is the purely algebraic direct sum of the isotypical components for the action of G on $C(X)$ [10, Proposition 2.2]; and (2) each isotypical component for the action of G on $C(Y \times G)$ is of the form $C(Y) \otimes V$, where V is an isotypical component for the action of G on $C(G)$.

As $\mathcal{O}(G)$ is a Hopf algebra, the dual of the homeomorphism

$$F_G: G \times G \ni (g_1, g_2) \mapsto (g_1, g_1 g_2) \in G \times G$$

and the dual of its inverse F_G^{-1} restrict and corestrict respectively to

$$\begin{aligned}
\text{can}_G: \mathcal{O}(G) \otimes \mathcal{O}(G) \ni T &\mapsto T \circ F_G \in \mathcal{O}(G) \otimes \mathcal{O}(G), \\
\text{can}_G^{-1}: \mathcal{O}(G) \otimes \mathcal{O}(G) \ni T &\mapsto T \circ F_G^{-1} \in \mathcal{O}(G) \otimes \mathcal{O}(G).
\end{aligned}$$

Granted the identification (5), we now obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{P}_G(Y \times G) \otimes_{C(Y)} \mathcal{P}_G(Y \times G) & \xrightarrow{\text{can}} & \mathcal{P}_G(Y \times G) \otimes \mathcal{O}(G) \\
\downarrow = & & \downarrow = \\
C(Y) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) & \xrightarrow{\text{id} \otimes \text{can}_G} & C(Y) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G).
\end{array}$$

Hence the bijectivity of can_G implies the bijectivity of can .

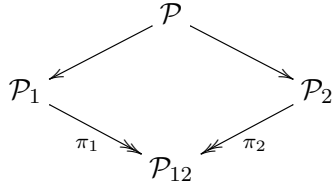
3 Appendix

In this appendix, we observe that Step 2.5, i.e. reduction to the trivial-bundle case, is implied by the following general results:

Lemma 2. *Let (H, Δ) be a compact quantum group acting on a unital C^* -algebra A . The assignment $A \mapsto \mathcal{P}_H(A)$ of the Peter–Weyl algebra to a C^* -algebra yields a functor from the category with objects being unital C^* -algebras with an H -coaction (and morphisms being equivariant unital $*$ -homomorphisms) to the category whose objects are $\mathcal{O}(H)$ -comodule algebras (and whose morphisms are colinear algebra homomorphisms). Furthermore, this functor commutes with all (equivariant) pullbacks, that is,*

$$\mathcal{P}_H\left(A \times_B C\right) = \mathcal{P}_H(A) \times_{\mathcal{P}_H(B)} \mathcal{P}_H(C).$$

Lemma 3 (Lemma 3.2 in [5]). *Let \mathcal{H} be a Hopf algebra with bijective antipode, and let*



be the pullback diagram of surjective right \mathcal{H} -comodule algebra homomorphisms. Then \mathcal{P} is principal if and only if \mathcal{P}_1 and \mathcal{P}_2 are principal¹.

Proving the first lemma is straightforward, and the second lemma is the highlight of [5]. For the case considered in this paper (a compact Hausdorff group G acting continuously on a compact Hausdorff space X), we have $A = C(X)$, $H = C(G)$, $\mathcal{O}(H) = \mathcal{O}(G)$, $\mathcal{P}_H(A) = \mathcal{P}_G(X)$, and the condition of being “principal” is equivalent to the PWG-condition. Thus these two lemmas combined with standard induction yield an alternative proof of Step 2.5.

Moreover, the theorem of this paper is a special case of a much more general theorem about compact quantum groups acting on unital C^* -algebras [1]. However, the proof of the general theorem is nonlocal.

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References

- [1] Baum P.F., De Commer K., Hajac P.M., Free actions of compact quantum groups on unital C^* -algebras, [arXiv:1304.2812](#).
- [2] Baum P.F., Hajac P.M., Matthes R., Szymański W., Noncommutative geometry approach to principal and associated bundles, in Quantum Symmetry in Noncommutative Geometry, to appear, [math.DG/0701033](#).
- [3] Brzeziński T., Hajac P.M., The Chern–Galois character, *C. R. Math. Acad. Sci. Paris* **338** (2004), 113–116, [math.KT/0306436](#).
- [4] Gleason A.M., Spaces with a compact Lie group of transformations, *Proc. Amer. Math. Soc.* **1** (1950), 35–43.

¹For the definition of “principal” see [3] and [5, Definition 2.3].

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- [5] Hajac P.M., Krähmer U., Matthes R., Zieliński B., Piecewise principal comodule algebras, *J. Noncommut. Geom.* **5** (2011), 591–614, [arXiv:0707.1344](#).
 - [6] Loomis L.H., An introduction to abstract harmonic analysis, D. Van Nostrand Company, Inc., Toronto – New York – London, 1953.
 - [7] Mostow G.D., Cohomology of topological groups and solvmanifolds, *Ann. of Math.* **73** (1961), 20–48.
 - [8] Podleś P., Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups, *Comm. Math. Phys.* **170** (1995), 1–20, [hep-th/9402069](#).
 - [9] Schneider H.-J., Principal homogeneous spaces for arbitrary Hopf algebras, *Israel J. Math.* **72** (1990), 167–195.
 - [10] Soltan P.M., On actions of compact quantum groups, *Illinois J. Math.* **55** (2011), 953–962, [arXiv:1003.5526](#).